

# The MOND paradigm

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## ABSTRACT

I review briefly different aspects of the MOND paradigm, with emphasis on phenomenology, epitomized here by many MOND laws of galactic motion—analogue to Kepler’s laws of planetary motion. I then comment on the possible roots of MOND in cosmology, possibly the deepest and most far reaching aspect of MOND. This is followed by a succinct account of existing underlying theories. I also reflect on the implications of MOND’s successes for the dark matter (DM) paradigm: MOND predictions imply that baryons alone accurately determine the full field of each and every individual galactic object. This conflicts with the expectations in the DM paradigm because of the haphazard formation and evolution of galactic objects and the very different influences that baryons and DM are subject to during the evolution, as evidenced, e.g., by the very small baryon-to-DM fraction in galaxies (compared with the cosmic value). All this should disabuse DM advocates of the thought that DM will someday be able to reproduce MOND: it is inconceivable that the modicum of baryons left over in galaxies can be made to determine everything if a much heavier DM component is present.

## 1. Introduction: the MOND paradigm

MOND is an alternative paradigm to Newtonian dynamics, whose original motivation was to explain the mass discrepancies in galactic systems without invoking dark matter (DM) (Milgrom 1983a). It constitutes a modification of dynamics in the limit of low accelerations that rests on the following basic assumptions: (i) There appears in physics a new constant,  $a_0$ , with the dimensions of acceleration. (ii) Taking the formal limit  $a_0 \rightarrow 0$  in all the equations of physics restores the equations of classical (pre-MOND) dynamics. (iii) For purely gravitational systems, the opposite, deep-MOND limit,  $a_0 \rightarrow \infty$ , gives limiting equations of motion that can be written in a form where the constants  $a_0$  and  $G$ , and all masses in the problem,  $m_i$ , appear only in the product  $m_i G a_0 = m_i / \mu_0$ , where  $\mu_0 \equiv (G a_0)^{-1}$  (Milgrom 2005)<sup>1</sup>. This last fiat reproduces the desired MOND phenomenology for purely gravitational systems. A MOND theory is one that incorporates the above tenets in the nonrelativistic regime.

Since all our knowledge of MOND comes, at present, from the study of purely gravitational systems (galactic systems, the solar system, etc.) it is still an open question how exactly to extend the third MOND tenet to systems involving arbitrary interactions. One possibility is to require that for  $a_0 \rightarrow \infty$ , the limiting equations of motion can be brought to a form where  $G$ ,  $a_0$ , and  $m_i$  appear only as  $G a_0^2 = a_0 / \mu_0$  and  $m_i / a_0$ . This requirement will automatically cause  $a_0$ ,  $G$ , and  $m_i$  to appear as  $m_i / \mu_0$  in the deep MOND limit of purely gravitational systems, as in such cases  $G$  and  $m_i$  always appear in equations as  $G m_i$ . Such a general

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<sup>1</sup>By this I mean that starting with equations that involve (including in derivatives)  $\mathbf{r}$ ,  $t$ ,  $G$ ,  $a_0$ ,  $m_i$ , and gravitational field degrees of freedom, we can rewrite them, possibly by redefining the gravitational field, so that only  $\mathbf{r}$ ,  $t$ ,  $m_i / \mu_0$ , and the gravitational field appear.

requirement would also replace Newton’s second law  $\mathbf{F} = m\mathbf{a}$  by  $\mathbf{F} = m\mathbf{Q}/a_0$  in the deep MOND regime, where  $\mathbf{Q}$  is some functional with dimensions of acceleration squared that does not depend on  $a_0$ .

Detailed reviews of phenomenological and theoretical aspects of the paradigm can be found, for instance, in Sanders and McGaugh (2002), and, more recently, in Scarpa (2006) and in Bekenstein (2006).

It follows from the third tenet that an underlying MOND theory must be nonlinear in the sense that an acceleration of a test particle due to a combination of several fields is not simply the sum of the accelerations produced by the individual fields. Take, as an example, the purely gravitational case where we modify the equations for the gravitational field. Linearity would mean that in the nonrelativistic limit of the theory the acceleration of a test particle at position  $\mathbf{r}$  in the field of  $N$  masses  $m_i$  at positions  $\mathbf{r}_i$  is given by

$$\mathbf{a} = \sum_{i=1}^N m_i q_i(G, a_0, \mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_N). \quad (1)$$

The third assumption says that in the deep MOND case  $q_i \propto \mu_0^{-1}$ , but this is dimensionally impossible. Clearly, the acceleration produced even by a single point cannot be linear in its mass, as we shall see explicitly below.

It also follows from the third tenet, and the assumption that  $a_0$  is the only new dimensioned constant, that the deep MOND limit of any theory must satisfy the following scaling laws for purely gravitational systems: In general, on dimensional grounds, all physics must remain the same under a change of units of length  $\ell \rightarrow \lambda\ell$ , of time  $t \rightarrow \lambda t$ , and no change in mass unit  $m \rightarrow m$ . Under these we have  $a_0 \rightarrow \lambda^{-1}a_0$  and  $G \rightarrow \lambda G$ ; so, the constant  $\mu_0$ , which alone appears in the limiting theory, is invariant under the scaling<sup>2</sup>. This tells us that we are, in fact, exempt from scaling the constants of the theory when we scale  $(\mathbf{r}, t) \rightarrow \lambda(\mathbf{r}, t)$ . The theory is thus invariant under this scaling; namely, if a certain configuration is a solution of the equations, so is the scaled configuration. Specific theories may have even higher symmetries; for example, the above scaling property is only part of the conformal invariance of the deep MOND limit for the particular MOND theory of Bekenstein & Milgrom (1984), as found in Milgrom (1997).

As a corollary of the scaling invariance we have: if  $\mathbf{r}(t)$  is a trajectory of a point body in a configuration of masses  $m_i$  at positions  $\mathbf{r}_i(t)$  (which can be taken as fixed, for example), then  $\hat{\mathbf{r}}(t) = \lambda\mathbf{r}(t/\lambda)$  is a trajectory for the configuration where  $m_i$  are at  $\lambda\mathbf{r}_i(t/\lambda)$ , and the velocities on that trajectory are  $\hat{\mathbf{V}}(t) = \mathbf{V}(t/\lambda)$ . (An extended mass changes its size and density such that the total mass remains the same. A point mass remains a point mass of the same value.)

Since  $\mu_0$  has dimensions of  $mt^4/\ell^4$ , another scaling under which it is invariant is  $m_i \rightarrow \lambda m_i$ ,  $\mathbf{r}_i \rightarrow \mathbf{r}_i$ ,  $t \rightarrow \lambda^{-1/4}t$ . (The most general scaling allowed is  $m \rightarrow \lambda m$ ,  $\mathbf{r}_i \rightarrow \kappa\mathbf{r}_i$ ,  $t \rightarrow \kappa\lambda^{-1/4}t$ .) This means that for a purely gravitational system deep in the MOND regime, scaling all the masses leaves all trajectory paths the same but the bodies traverse them with all velocities scaling as  $m^{1/4}$ ; accelerations then scale as  $m^{1/2}$ .

It is enlightening to draw an analogy between the role of  $a_0$  in MOND with the role of  $\hbar$  in quantum physics, or that of  $c$  in relativity. These constants, each in its own realm, mark the boundary between the classical and modified regimes; so formally pushing these boundaries to the appropriate limits ( $c \rightarrow \infty$ ,  $\hbar \rightarrow 0$ ,  $a_0 \rightarrow 0$ ) one restores the corresponding classical theory (for quantum theory, in some weak sense). In addition, these constants enter strongly the physics in the modified regime where they feature in various

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<sup>2</sup>Other dimensioned quantities, such as the gravitational potential, have to be scaled appropriately. Note that velocities are invariant.

phenomenological relations. For example,  $\hbar$  appears in the black body spectrum, the photoelectric effect, atomic spectra, and in the quantum Hall effect. The speed of light appears in the Doppler effect, in the mass vs. velocity relation, and in the radius of the Schwarzschild horizon. Without the respective, underlying theory, these disparate phenomena would appear totally unrelated, and the appearance of the same numerical constant in all of them would constitute a great mystery. The MOND paradigm similarly predicts a number of laws related to galactic motion, some of which are qualitative, but many of which are quantitative and involve  $a_0$ . Since they appear to be obeyed by nature it should indeed be a great mystery why that should be so without the underlying MOND paradigm (i.e. with Newtonian dynamics plus DM). I now discuss some of these predictions in some detail.

## 2. MOND laws of galactic dynamics

A MOND theory based on the above basic premises should predict everything about the acceleration field of an object, such as a galaxy, based on the baryon distribution alone. But even before looking at detailed predictions for individual systems, it is possible and useful to distil a number of corollaries that follow essentially from the basic premises themselves. This helps focus attention on some unifying, general laws, which may be likened to Kepler’s laws of planetary motion. MOND, of course, predicts more than just these laws.

I list some such relations below and explain how they come about. It is important to realize that some of these actually contradict the predictions of CDM, and that those that do not are independent in the context of DM. They are so in the sense that one can construct galaxy models with baryons and DM that satisfy any set of these predictions but not any of the others. So, in the context of DM each would require a separate explanation. The MOND predictions concerning the mass discrepancies in galactic systems depend only on the present day baryon distribution. In contrast, the expected discrepancies; i.e., the relative quantities and distributions of baryons and DM in such systems depend strongly on their unknown (and unknowable) formation history, a point which I expand on in section 8.

In deriving the predictions below I assume that the theory involves only  $a_0$  as a new dimensioned constant.

1. The orbital speed around an arbitrary isolated mass becomes independent of the radius of the orbit in the limit of very large radii. This means, in particular, that the rotational velocity on a circular orbit becomes independent of the orbital radius at large radii—asymptotic flatness of rotation curves:  $V(r) \rightarrow V_\infty$ . This quantitative behavior was the cornerstone on which MOND was built; so it was introduced by hand, based on anecdotal evidence existing at the time. But once taken as axiom of MOND it has become a binding prediction. It follows straightforwardly from the above length-time scaling property of the deep MOND limit<sup>3</sup>; but it is worthwhile seeing an explicit derivation: On dimensional grounds, the acceleration of a test particle on an arbitrary orbit around a point mass,  $M$ , when it is at a distance  $r$  on the orbit, has to scale with  $M$  and  $r$  as

$$a \sim \frac{MG}{r^2} \nu \left( \mathbf{q}, \frac{MG}{r^2 a_0} \right), \quad (2)$$

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<sup>3</sup>Just as the analogue third Kepler law for Newtonian dynamics,  $V(r) \propto r^{-1/2}$ , follows from the fact that masses and  $G$  always appear in the combination  $MG$ , which is invariant under  $\mathbf{r} \rightarrow \lambda \mathbf{r}$ ,  $t \rightarrow \lambda^{3/2} t$ , under which  $V \rightarrow \lambda^{-1/2} V$ .

where  $\nu(\mathbf{q}, y)$  can depend, through some dimensionless parameters  $\mathbf{q}$ , on the geometry of the orbit and on the position on the orbit<sup>4</sup>. From the basic premises of MOND we must have:

$$\nu(\mathbf{q}, y) \approx \begin{cases} 1 & : y \gg 1 \\ \eta(\mathbf{q})y^{-1/2} & : y \ll 1 \end{cases} . \quad (3)$$

This means that at large  $r$  we have  $a = \eta(\mathbf{q})(MGa_0)^{1/2}/r$ . This, in turn, means that the orbital speed is given by

$$V(M, r, \mathbf{q}) = \lambda(\mathbf{q})(MGa_0)^{1/4} = \lambda(\mathbf{q})(M/\mu_0)^{1/4}, \quad (4)$$

and is thus invariant to scaling of the orbit. This holds for any MOND theory (be it modified gravity or modified inertia—see below) and for any orbit. For circular orbits in an axisymmetric potential  $\lambda$  has to be constant as it cannot depend on orbital phase. The value of  $a_0$  is normalized so that for circular orbits  $\lambda = 1$ .

The function  $\nu(\mathbf{q}, y)$  is one example of the appearance of a so called interpolating function in MOND [the name is usually reserved for the function  $\mu(x)$  related to  $\nu$  by  $\mu(x) = i(x)/x$ , where  $i(x)$  is the inverse of  $y\nu(y)$ ]. In some formulations of MOND (as in modified gravity—see below) there is a single interpolating function that appears in the theory. In other formulations there isn't a unique one (as, for example here, where  $\nu$  can depend on  $\mathbf{q}$ , in general). In a given context it interpolates between the known behaviors in the deep MOND regime and the classical (pre MOND) regime.

2. MOND predicts a relation between the total mass of a body and the asymptotic circular velocity around it. It follows from eq.(4) that  $V_\infty^4 = MGa_0$  (Milgrom 1983b). This can be tested directly for disc galaxies (see confirming analysis in McGaugh et al. 2000, McGaugh 2005a,b). This predicted mass-asymptotic-speed relation is in the basis of the traditional, empirical Tully-Fisher (TF) relation between galaxy luminosity and rotational speed, from which, however, it differs in essence. One finds in the literature various empirical plots that are termed TF relations, between some luminosity measure and some velocity measure. Many of these plots are not useful for comparison with theoretical predictions. MOND dictates exactly what should be plotted: the total baryonic mass against the rotational velocity on the flat part of the rotation curve—aka the baryonic Tully-Fisher (TF) relation. Using, as in the original TF relation, the luminosity as a mass measure will not do; at best, it represents only the stellar mass, leaving out the gas mass. This is clearly demonstrated in Fig.1 (see also Milgrom & Braun 1988) where a tight relation is followed only for high mass galaxies, which are dominated by stellar mass. The same relation holds for lower mass galaxies, which are gas rich, only if we include the gas mass. Note also that the use of a velocity measure that is heavily weighted by smaller radii (e.g. optical velocity measures) artificially distorts the mass-velocity plot, introducing artificial scattering and biasing the slope of the relation: For low mass galaxies the inner velocities are, typically, smaller than the asymptotic ones, while for high mass galaxies the opposite is true.

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<sup>4</sup>I assume implicitly that for a mass distribution of total mass  $M$  that is bounded in a volume of size  $R$ , all that enters the motion of a test particle at large radii ( $r \gg R$ ) is  $M$ , while the exact spatial distribution of the mass is immaterial. In other words, I assume that the motion of a test particle in the field of a point mass is independent of the way the limit of the point mass is taken. This is certainly true for modified inertia theories in which the gravitational field equation is the standard one. For modified gravity theories it seems an obvious assumption, but, in principle, it has to be checked (it does hold for all the theories studied to date).

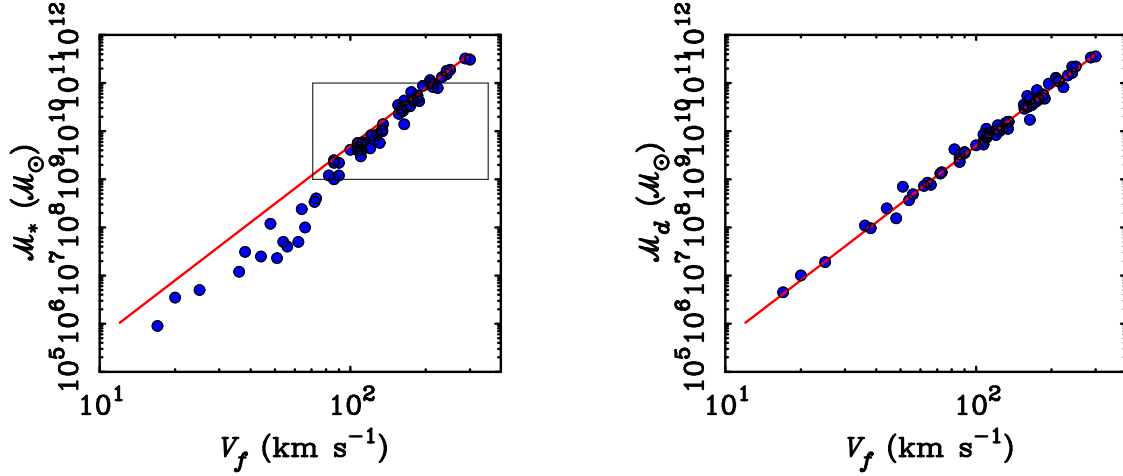


Fig. 1.— Galaxy mass plotted against the rotation curve plateau velocity. Left: analog of the traditional Tully-Fisher plot with mass in stars only. Right: The total mass including that of gas. The solid line has the log-log slope of 4, predicted by MOND, and is not a fit (McGaugh 2005b) (the small rectangle shows where past analysis had concentrated).

3. In a disc galaxy, whose rotation curve is  $V(r)$ , that has high central accelerations ( $V^2/r > a_0$  in the inner regions), the mass discrepancy appears always around the radius where  $V^2/r = a_0$ . In galaxies whose central acceleration is below  $a_0$  (low surface brightness galaxies—LSBs) there should appear a discrepancy at all radii (Milgrom 1983b, and for a confirmation see, e.g., McGaugh 2006).
4. For a concentrated mass,  $M$ , well within its transition radius,  $r_t \equiv (MG/a_0)^{1/2}$ ,  $r_t$  plays a special role (somewhat akin to that of the Schwarzschild radius in General Relativity) since the dynamics changes its behavior as we cross from smaller to larger radii. For example a shell of phantom DM may appear around this radius (Milgrom & Sanders 2008).
5. Isothermal spheres have mean surface densities  $\bar{\Sigma} \lesssim \Sigma_0 \equiv a_0/G$  (Milgrom 1984) underlying the observed Fish law for quasi-isothermal stellar systems such as elliptical galaxies (see discussion in Sanders and McGaugh 2002). This follows from the fact that Newtonian, self gravitating, isothermal spheres (ISs) have an enclosed mass that increases linearly with radius, and they thus have an infinite total mass. The only way to avoid this is via a mass cutoff provided by MOND, which can only be felt around  $r_t$ . But, by definition, the mean surface density within  $r_t$  is  $\sim \Sigma_0$ . Unlike Newtonian ISs, MOND ISs have, indeed, a finite total mass. They can have arbitrarily small mean surface densities.
6. For spheroidal systems a mass-velocity-dispersion relation  $\sigma^4 \sim MGa_0$  is predicted under some circumstances. According to MOND, this is the fact underlying the observed Faber-Jackson relation for elliptical galaxies, which are approximately isothermal spheres (Milgrom 1984). For instance, this relation holds approximately for all isothermal spheres having a constant velocity dispersion  $\sigma$  and constant velocity anisotropy ratio  $\beta$  (Milgrom 1984). Such spheres are also characterized by a mass  $M$ , and some mean radius,  $R$ . On dimensional grounds we have, in a given MOND theory:  $\sigma^4 = f\left(\beta, \frac{MG}{R^2a_0}\right) MGa_0$ . From the basic premises of MOND, when  $MG/R^2a_0$  is small compared with 1,  $f$  becomes independent of this quantity. Prediction 5 says that this ratio cannot be large compared with 1. These give

$\sigma^4 \sim \hat{f}(\beta)MGa_0$  ( $\hat{f}$  depends weakly on  $MG/Ra_0$ ). (See Milgrom 1984 for detailed calculations in a particular MOND theory). In the same MOND theory Milgrom (1994b) derived a general relation for arbitrary, stationary, self gravitating, many-particle systems (not necessarily isothermal, or spheroidal) in the deep MOND regime, of the form  $\sigma^4 = (4/9)MGa_0$ , where  $\sigma$  is the 3-D rms velocity dispersion. Sanders (2000) discussed a generalization of isothermal spheres that better fit observed ellipticals.

Note the difference between this relation and the mass-velocity relation in law 2. The latter holds for all systems and is exact, but involves the large radius asymptotic rotational speed; the former is limited to either isothermal spheres or to low acceleration systems (e.g., it does not apply to stars, which are neither), is only approximate even for these, and involves the bulk velocities.

7. There is a difference in the dynamics, and hence in the stability properties, of discs with mean surface density  $\bar{\Sigma} \lesssim a_0/G$  and  $a_0/G \lesssim \bar{\Sigma}$  (Milgrom 1989a, Brada & Milgrom 1999b, Tiret & Combes 2007a,b).
8. The excess acceleration that MOND produces over Newtonian dynamics, for a given mass distribution, cannot much exceed  $a_0$  (Brada & Milgrom 1999a). This simply follows from the fact that MOND differs from Newtonian dynamics only when the accelerations are around or below  $a_0$ . Put in terms of DM this MOND prediction would imply that the acceleration produced by a DM halo alone can never much exceed  $a_0$ , according to MOND. There is no known reason for this to hold in the context of the DM paradigm. This prediction was confirmed by Milgrom & Sanders (2005) for a sample of disc galaxies.
9. An external acceleration field,  $g_e$ , enters the internal dynamics of a system imbedded in it. For example, if the system's intrinsic acceleration is smaller than  $g_e$ , and both are smaller than  $a_0$ , the internal dynamics is quasi-Newtonian with an effective gravitational constant  $Ga_0/g_e$  (Milgrom 1983a, 1986a, Bekenstein & Milgrom 1984). This was applied to various astrophysical systems such as dwarf spheroidal galaxies in the field of a mother galaxy, warp induction by a companion, escape speed from a galaxy, departure from asymptotic flatness of the rotation curve, and others (see, e.g., Brada & Milgrom 2000a,b, Famaey, Bruneton, & Zhao 2007, Angus & McGaugh 2007, and Wu et al. 2007). This external field effect (EFE) follows from the inherent nonlinearity of MOND. It appears in somewhat different ways in all versions of MOND studies to date, but I am not sure that it is a general consequence of the basic premises alone.

In some applications the external field effect can be mimicked by a cutoff in a DM halo; but, others of its consequences violate the strong equivalence principle, and are thus not reproducible with DM; e.g., the time dependent, non-tidal effects on the structure and internal kinematic of dwarf spheroidals that fall in the field of a mother galaxy, and the induction of warps discussed by Brada & Milgrom (2000a,b).

10. The nonlinearity of MOND also leads to a breakdown of the thin lens approximation: Two different mass distributions having the same projected surface density distributions on the sky, do not produce the same lensing effect as is the case, approximately, in General Relativity (Mortlock and Turner 2001, Milgrom 2002a). For example, consider a chain of  $N$  equal, point masses  $m$  far apart from each other along the line of sight [much farther than their individual transition radius  $(mG/a_0)^{1/2}$ ], but closer together than the observer-lens and the lens-source distances. In standard dynamics they act as a single point mass  $Nm$ . In deep MOND the gravitational field scales as  $m^{1/2}$ , so the effect for a single mass  $Nm$  scales as  $(Nm)^{1/2}$ , while that of the chain scales as  $Nm^{1/2}$  (Milgrom 2002a). Some implications and applications of this are discussed, e.g., in Milgrom & Sanders (2008), and in Xu et al. (2007). This MOND law conflicts with the predictions of DM.

11. Disc galaxies are predicted to exhibit a disc mass discrepancy, as well as the spheroidal one that is found for any mass. In other words, when MOND is interpreted as DM we should deduce a disc component of DM as well as a spheroidal one (Milgrom 1983b, 2001). The reason: the dynamical surface density in a disc is deduced from the normal component of acceleration just outside it. Since the MOND prediction for this acceleration differs from the Newtonian value (and is generically larger) MOND predicts that Newtonian analysis will find a higher dynamical surface density than is observed. Unlike the spheroidal component, which extends to large radii, the disc component is confined to the baryonic disc: Where there is no baryonic disc there is no jump in the normal component of the acceleration and hence no mass discrepancy is predicted there. In LSB galaxies this phantom disc should be found everywhere in the baryonic disc; in HSB galaxies only where  $V^2/r \lesssim a_0$ . For thin discs, at radii where the radial acceleration dominates the perpendicular one, and  $V^2/r \ll a_0$ , the surface density of the phantom disc is  $\sim (V^2/ra_0)^{-1}$  times the baryon surface density in the disc. For results of analysis in support of this prediction see Milgrom (2001), Kalberla et al. (2007)<sup>5</sup>, and Sánchez-Salcedo, Saha, & Narayan (2007). This prediction contradicts the expectations for Cold dark matter, which, being dissipationless, forms only spheroidal halos, not discs.
12. A DM interpretation of MOND should give negative density of “dark matter” in some locations (Milgrom 1986b): The reason: there is no guarantee that the divergence of the MOND acceleration field is larger than  $4\pi G\rho$  everywhere, where  $\rho$  is the baryon density. The density distribution deduced from the Poisson equation may thus fall below the observed baryon source density, in places. If verified, this would directly conflict with DM.

### 3. Galaxy rotation curves in MOND

The quintessential MOND result, above and beyond the preceding general laws, is the prediction of the rotation curves (RC) of individual disc galaxy, based only on the observed (baryonic) mass distribution. The force of this prediction was clear from the beginning; but, interestingly, it took some five years after the advent of MOND for the first RC analysis to be performed as described in Kent (1987) and the amending sequel by Milgrom (1988). Meaningful analysis had had to await the appearance of extended RCs afforded by HI observations. Many such analyses followed (Milgrom & Braun 1988, Lake 1989 and its rebuttal in Milgrom 1991, Begeman Broeils & Sanders 1991, Broeils 1992, Morishima & Saio 1995, Sanders 1996, Moriondo Giovanardi & Hunt 1998, Sanders and Verheijen 1998, de Blok & McGaugh 1998, Bottema et

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<sup>5</sup>Interestingly, Kalberla et al. find that in order to reproduce the observed flaring data of the Milky Way disc one needs, in addition to a spheroidal halo of DM, a disc component and also a disc-like ring of DM centered at about 16 kpc. I find in Milgrom (2008b) that for most forms of  $\mu(x)$ , the fictitious DM disc predicted by MOND has a maximum in its surface density at a radius of the order of the transition radius, and could thus mimic a disc-plus-ring component of DM as is found in the analysis of Kalberla et al.. This disc-like ring predicted by MOND is analogous to the phantom shell of “DM” predicted for concentrated masses, as discussed in Milgrom & Sanders (2008) (see law number 4 above). I show some such predicted phantom discs in Fig. 2 for a heuristic model of the Milky Way. The model consists of a thin exponential disc of scale length 0.3 in units of the transition radius, and a de Vaucouleur sphere with effective radius 0.1 in these units. The disc-to-bulge mass ratio is 1:0.3. (For a MW mass of  $10^{11}M_\odot$ , or asymptotic rotational speed of  $200 \text{ km s}^{-1}$ , the transition radius is at  $\approx 11 \text{ kpc}$ , assuming  $a_0 = 1.2 \times 10^{-8} \text{ cm s}^{-2}$ .) Sánchez-Salcedo et al. use interpolating functions of the forms  $\mu_1$  and  $\mu_2$  (in the nomenclature of Milgrom & Sanders 2008), which as we see in Fig. 2 are not completely in line with the deductions of Kalberla et al., as these forms predict no surface density peak or one at smaller radii. Indeed the MOND results of Sánchez-Salcedo et al., while satisfactory, are somewhat lacking, and may do better for other choices of  $\mu$  that give a surface density peak at larger radii.

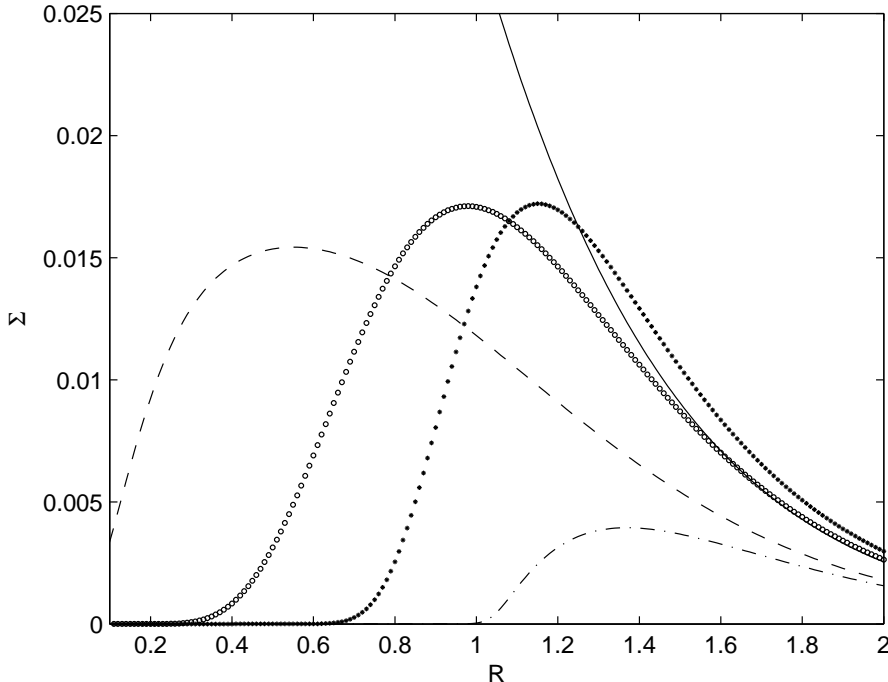


Fig. 2.— The MOND prediction of the fictitious DM disc surface density in units of  $\Sigma_0 = a_0/G$ , as a function of galactocentric radius in units of the transition radius, for a heuristic model of the Milky Way. Results for five interpolating functions are shown using the designation in Milgrom & Sanders (2008):  $\mu_1$  (solid),  $\mu_2$  (dashed),  $\bar{\mu}_{1.5}$  (circles),  $\bar{\mu}_3$  (stars), and  $\mu_{40}$  (dot-dash).

al. 2002, Begum & Chengalur 2004, Gentile et al. 2004, 2007a, 2007b, Corbelli & Salucci 2007, Milgrom & Sanders 2007, Barnes Kosowsky & Sellwood 2007, Sanders & Noordermeer 2007, Milgrom 2007). In all these analyses one has used the MOND relation

$$\mu(V^2/ra_0)V^2/r = g_N, \quad (5)$$

where  $\mu(x)$  is a MOND interpolating function, and  $g_N$  is the Newtonian acceleration calculated from the observed baryon distribution. Such a relation is an exact result in modified inertia theories for circular motion in axisymmetric potentials, as applies here, and is approximate for modified gravity theories.

Most RC analyses in the past have involved medium- to low-acceleration galaxies, for which the MOND prediction is not sensitive to the exact form of the interpolating function, as long as it satisfies the required small-arguments limit (see e.g., Milgrom & Sanders 2008). Analyses of HSB galaxies, which can constrain  $\mu(x)$ , have started in earnest only recently; for example in Famaey & Binney (2005), Zhao & Famaey (2006), Sanders & Noordermeer (2007), and Milgrom & Sanders (2008).



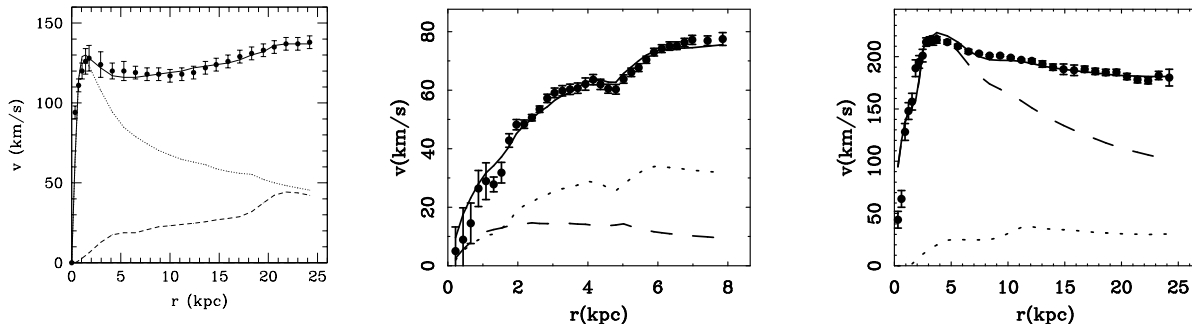


Fig. 3.— The observed and MOND rotation curves (in solid lines) for NGC 3657 (left), NGC 1560 (center), and NGC 2903 (right). The first from Sanders (2006a), the last two from Sanders and McGaugh (2002). Dotted and dashed lines are the Newtonian curves calculated for the different baryonic components (they add in quadrature to give the full Newtonian curve). For NGC 3657 the dotted line is for stars and dashed for the gas, with the reverse for the other two galaxies.

Figure 3 shows examples of MOND rotation curve analysis for three galaxies of very different types. In the center is NGC 1560, a very low acceleration, gas dominated galaxy, that has a rising RC within the observed baryons. To the left is NGC 3657, an intermediate case with similar contributions from the stars and gas. To the right is NGC 2903, a high acceleration galaxy dominated by stars, with a declining RC (after the inevitable initial rise). For NGC 1560, the MOND rotation curve is practically a prediction: since stellar mass contributes very little the  $M/L$  fit parameter gives hardly any leverage. In addition, since the accelerations are very small everywhere, the exact form of the interpolating function is immaterial. The same is true for quite a number of low surface galaxies of this type. For the other two galaxies, and the many others like them, the fit  $M/L$  parameter does have leverage, but a very limited one: We can view it, for example, as determined by the very inner part of the RC, so that the rest of the RC shape and amplitude becomes an exact, unavoidable prediction of MOND. To boot, the resulting best fit  $M/L$  values are not completely free; they have to fall in the right ballpark dictated by population synthesis, as, indeed, they do (e.g. Sanders & Verheijen 1998). Figs. 4, 5, 6 are mosaics of additional MOND RC results.

#### 4. Round systems

MOND analysis of globular clusters, dwarf spheroidals, elliptical galaxies, galaxy groups, galaxy cluster, and even one case of a super cluster have been considered (see reviews in Sanders & McGaugh 2002, and in Scarpa 2006). Here I shall concentrate only on galaxy clusters, which have not yet fully conformed to MOND’s predictions.

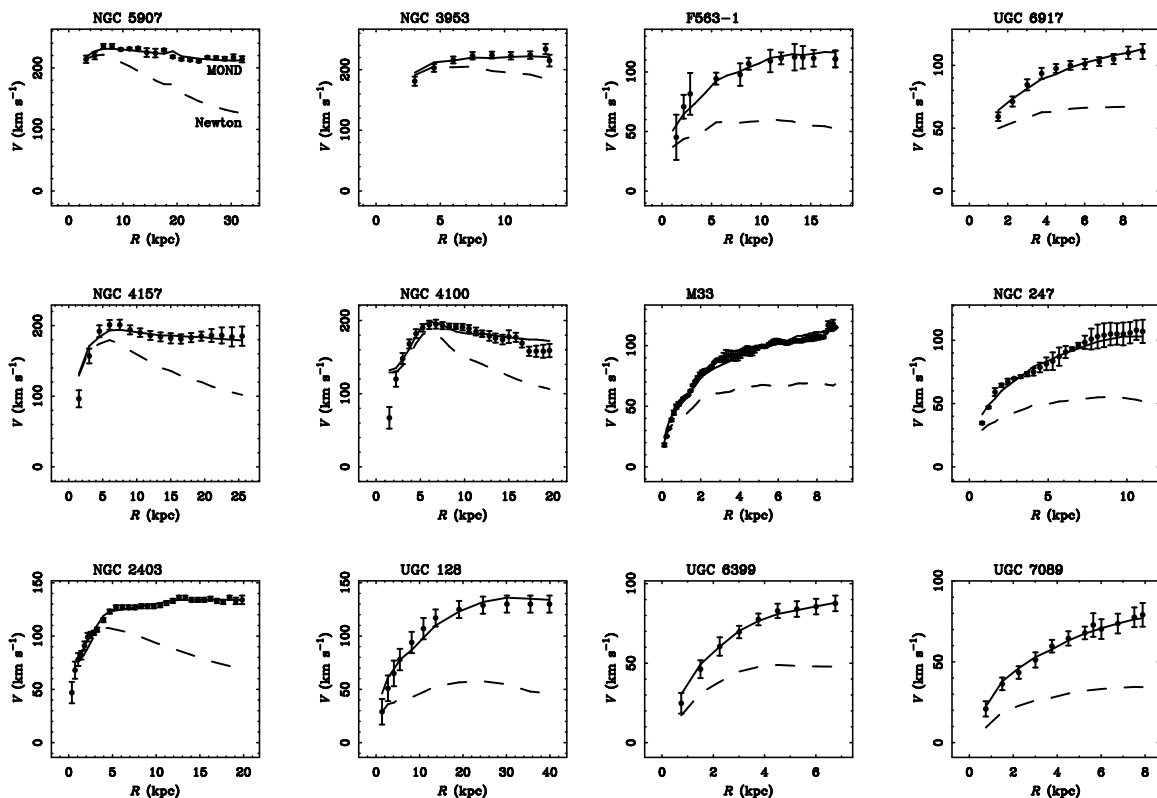


Fig. 4.— Additional MOND rotation curves from McGaugh (private communication). Dashed lines are the Newtonian curves, the solid lines the MOND curves.

#### 4.1. Cluster dark matter in light of MOND

As was realized years ago, MOND does not yet fully account for the mass discrepancy in galaxy clusters. The first MOND analysis of clusters (Milgrom 1983c) lowered the deduced  $M/L$  values by a large factor ( $\sim 5 - 10$ ) compared with Newtonian values. But, it still left some clusters with values of a few tens solar units, compared with several hundreds solar units gotten from Newtonian analysis. At the time it was wrongly thought that stars exhaust the baryonic budget of clusters; so  $M/L$  was taken to represent the mass discrepancy. In Milgrom (1983c) I speculated that the x-rays then known to emanate from clusters may bespeak the presence of large quantities of hot gas that will remove much of the remaining discrepancy. This has been largely vindicated by the identification and weighing of the hot intracluster gas. Reckoning with the gas reduces the discrepancy, in both MOND and Newtonian dynamics, by a factor  $\sim 5 - 10$ , but this is still not enough. Studies based on gas dynamics and on lensing have helped pinpoint the remaining discrepancy in MOND, as described by The & White (1988), Gerbal et al. (1992), Sanders (1994,1999,2003), Aguirre et al. (2001), Pointecouteau & Silk (2005), Angus Famaey & Buote (2007), and Takahashi & Chiba 2007. This remaining discrepancy has to be attributed to yet undetected matter, likely in some baryonic form (hereafter, CBDM—for cluster baryonic DM). Other suggestions involve massive neutrinos (Sanders 2003, 2007).

The following rough picture emerges regarding the distribution of the CBDM: Inside a few hundred kiloparsecs of the cluster center MOND makes only a small correction; so, most of the discrepancy observed

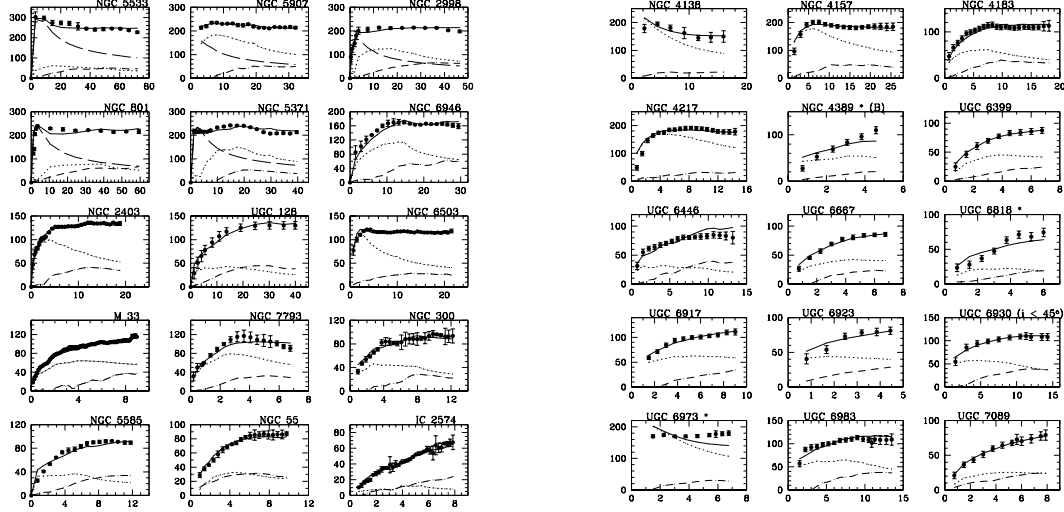


Fig. 5.— Additional MOND rotation curves from Sanders (1996) and de Blok & McGaugh (1998) (left) and from Sanders & Verheijen (1998) (right). (MOND curves in solid; stellar disc Newtonian curves in dotted; gas in dot-dash; and stellar bulge in long dashed.)

there must be due to CBDM. The ratio,  $\lambda$ , of accumulated CBDM to visible baryons there is  $\sim 10 - 50$ . The results of Sanders (1999) scaled to  $H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$  correspond roughly to  $\lambda = 2$  at 1 Mpc. (Pointecouteau & Silk 2005 claim higher values. But after correcting for two oversights on their part their results are consistent with those of Sanders.) The ratio  $\lambda$  decreases continuously with radius (see, e.g., the small-sample study of Angus, Famaey & Buote 2007). At the largest radii analyzed the gas mass still increases faster than the MOND dynamical mass meaning that  $\lambda$  is still decreasing there. We can extrapolate to higher radii and conclude that for the cluster as a whole  $\lambda$  is about 1 or even smaller. McGaugh (2007) reaches a similar conclusion based on extrapolating the mass-velocity relation from galaxies to clusters. In summary: in clusters at large the total mass in CBDM is comparable to that in the baryons already observed, but the CBDM is rather more centrally concentrated. So, another factor of two in the total baryon content of clusters should remove the remaining discrepancy in MOND. In this connection the historical lesson from the discovery of the hot gas might be of some value.

Note that the CBDM contributes only little to the total baryonic budget in the universe. Fukugita & Peebles (2004) estimate the total contribution of the hot gas in clusters to  $\Omega$  to be about 0.002. It follows from the above that the contribution of the CBDM is also only some 5 percent of the nucleosynthesis value.

If the CBDM is made of compact macroscopic objects—as is most likely—then, when two clusters collide, the CBDM will follow the galaxies in going through the collision center practically intact, whereas the gas components of the two clusters coalesce at the center. Clowe et al. (2006) have recently used weak lensing to map the mass distribution around a pair of colliding clusters and indeed found dark mass concentrations coincident with the galaxy concentrations to the sides of the gas agglomeration near the center of collision (See, however, Mahdavi et al. 2007 who claim a conflicting behavior). Such observations are consistent with what we already know about MOND dynamics of single clusters (Angus et al. 2007).

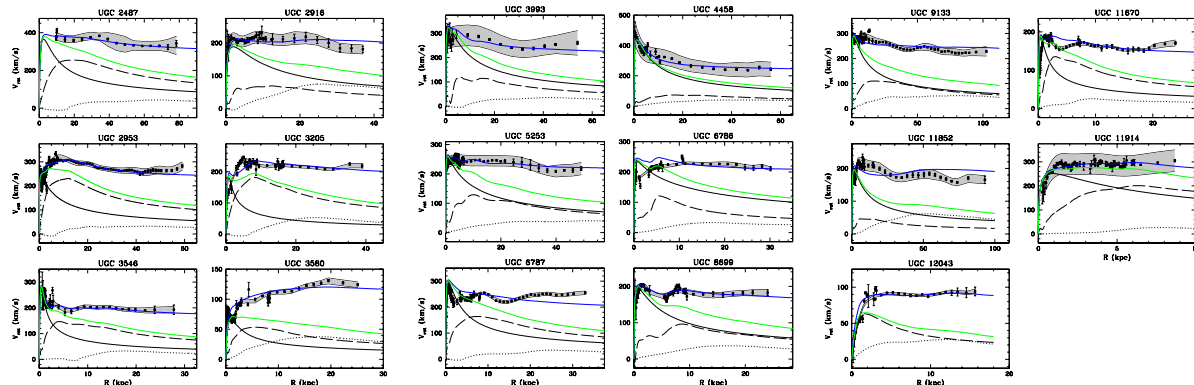


Fig. 6.— Additional MOND rotation curves from Sanders and Noordermeer (2007). The grey shaded bands give the allowed range due to inclination uncertainties. Thin black solid, dashed and dotted lines give the contributions from stellar bulge, disc and gas respectively. The thin green (grey) line gives the Newtonian sum of the individual components and the bold blue (grey) lines gives the total MOND rotation curve.

The presence of CBDM may in the end turn into a blessing as it might help alleviate the cooling flow puzzle in clusters (Milgrom 2008a).

Another interesting observation that concerns DM in galaxy clusters is brought to light by the recent claim by Jee et al. (2007) of a ring-like structure of DM surrounding the central part of the cluster Cl 0024+17. This was deduced on the basis of their weak lensing analysis that shows an enhancement (a bump) in the distribution of dynamical surface density, as reproduced in the left panel of our Fig. 7. This was brought up as a potential difficulty for MOND since, so it was claimed, no corresponding bump in the distribution of observed baryons is seen. If the presence of such a ring is confirmed it could be due to CBDM, which we know, in any case, has to be present (Famaey et al. 2007). However, interestingly, as shown by Milgrom & Sanders (2008), such a ring, with the observed characteristics, could also arise naturally as an artifact of MOND around the transition radius of the central mass (see prediction 4 above). A shell of phantom DM is predicted there, which appears as a projected ring. Verifying the ring as such a MOND effect would be very exciting as it will constitute a direct image of the transition region of MOND, analogous to the transition region of General Relativity as manifested by the formation of a horizon near  $MG/c^2$ . Figure 7 also shows, in the center and right panels, the predictions of MOND for the results of weak lensing around two rather simplistic mass distributions, with some favorable choices of the interpolating functions.

## 5. Solar System

The acceleration field of the sun is higher than  $a_0$  within  $\sim 8 \times 10^3$  astronomical units; so, by and large, MOND affects the motion of planets and spacecraft in the solar system only in its very high acceleration limit. In terms of the interpolating function, MOND predictions for such motion depend on the behavior of  $\mu(x)$  for  $x \gg 1$ . We have no knowledge of this limit from galaxy dynamics, which probes  $\mu$  to argument values of only a few. The fact that MOND effects on the planetary motions have not been observed has been used to constrain the high  $x$  behavior of  $\mu(x)$  ever since the advent of MOND; e.g., by Milgrom (1983a), Sereno & Jetzer (2006), Iorio (2007), and Bruneton & Esposito-Farese (2007). Solar system constraints on

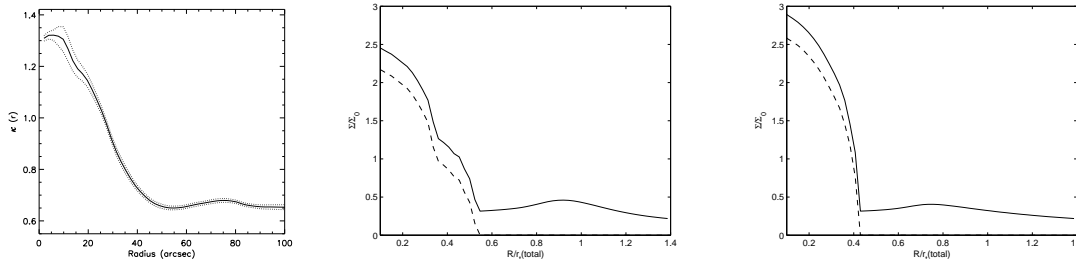


Fig. 7.— On the left: the deduced surface density distribution around the cluster Cl 0024+17 using weak lensing (Jee et al. 2007). The small bump is the alleged ring. The other two frames show the MOND predictions for very simplistic configurations produced by masses well contained within the radius of the bump taken from Milgrom & Sanders (2008).

relativistic theories in the high acceleration regime have been considered e.g., by Bekenstein (2004), Sanders (1997,2006b), and Bruneton & Esposito-Farese (2007). The potential for testing the low acceleration regime of MOND in special locations in the solar system, where accelerations almost cancel, was discussed by Bekenstein & Magueijo (2006); earth bound experiments were proposed by Ignatiev (2007).

An interesting possibility has been raised that the Pioneer anomaly (reviewed recently by Nieto & Anderson 2007) is due to MOND effects (e.g. discussion in Milgrom 2002a). The fact that the claimed anomalous acceleration is  $a_{anom} \approx 2\pi a_0$  is particularly suggestive. As explained in Milgrom (2002a), if indeed the anomaly is due to new physics it may point to the option of modified inertia, rather than modified gravity. This is because modified gravity predicts similar effects on the planets, contrary to what is measured (see also Tangen 2007). In modified inertia, where one modifies the equations of motion of particles, not the gravitational field, it is possible for particles to suffer very different MOND corrections at the same position, depending on their trajectory. So, it might be possible to construct theories that affect the Pioneer spacecraft on their straight, unbound trajectories without affecting as much the planets on their elliptical, bound orbits (see section 7.1.2 below).

## 6. $a_0$ and its cosmological significance

The constant  $a_0$  appears in many of the MOND laws of galactic motion listed in section 2. At first its value was, indeed, determined by appealing to some of these (2, 5, 6, and 11) as described in Milgrom (1983b). However, the most accurate handle on  $a_0$  comes today from rotation curve analysis. For example, Milgrom (1988) found  $a_0 \approx 1.3 \times 10^{-8} \text{ cm s}^{-2}$  from Kent’s reanalysis, and Begeman Broeils & Sanders 1991 found, with better data,  $a_0 \approx 1.2 \times 10^{-8} \text{ cm s}^{-2}$  (both used  $H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$ —the value of  $a_0$  depends on the assumed distances to galaxies).

Milgrom (1983a) observed that  $2\pi a_0 \approx cH_0$ . And, since we now know that “dark energy” makes up most of the closure density today, namely  $\Omega_\Lambda = \Lambda/3H_0^2 \sim 1$  we also have  $2\pi a_0 \approx c(\Lambda/3)^{1/2}$ .

With the aid of  $a_0$ ,  $G$ , and  $c$  we can construct a length scale  $\ell_0 \equiv c^2/a_0 \approx 10^{29} \text{ cm}$ , and a mass scale,  $M_0 \equiv \mu_0 c^4 \approx 6 \times 10^{23} M_\odot$ . This is similar to the emergence, in connection with quantum theory,

of the Planck length and the Planck mass, constructed from  $\hbar$ ,  $G$  and  $c$ . These set the boundary in the space of phenomena beyond which combined effects of strong gravity and quantum physics are expected. Similarly,  $\ell_0$  and  $M_0$  tell us where to expect MOND effects combined with strong gravity. Because  $\ell_0$  is of the order of the Hubble radius  $\ell_0 \approx 2\pi\ell_H$ , with  $\ell_H \equiv cH_0^{-1} \approx c(\Lambda/3)^{-1/2}$ , and  $M_0 \approx 2\pi M_U$ , with  $M_U \equiv c^3 G^{-1} (\Lambda/3)^{-1/2} \approx c^3 G^{-1} H_0^{-1}$ , combined effects are expected only for the universe at large, and thus there are no local black holes with surface accelerations in the MOND regime.

These coincidences may point to a very strong ties between MOND and cosmology connecting perhaps MOND and the “dark energy” effects: Either one is the result of the other or they are both induced by the same mechanism. If the parameter  $a_c \equiv M_U/\ell_H^2 = cH_0 \approx c(\Lambda/3)^{1/2}$  is somehow felt by local physics it may not be surprising that dynamics is different for acceleration above and below this value. It has been long suspected that local dynamics is strongly influenced by the universe at large, a-la Mach’s Principle, but MOND seems to be the first to supply a concrete evidence for such a connection. This may turn out to be the most fundamental implication of MOND, beyond its implied modification of Newtonian dynamics and General relativity, and beyond the elimination of DM.

One immediately notes that if the coincidence of  $a_0$  with cosmologically significant acceleration parameters is a causal one, it may hint that  $a_0$  varies with cosmic time (Milgrom 1989b, 2002a). For example, if we always have  $a_0 \sim cH_0$ , or if  $a_0$  is related to  $\Lambda$  and the latter changes with time, then  $a_0$  would follow suit. This is not necessarily the case since  $a_0$  could be related to  $\Lambda$  with the latter being constant. Interestingly, variations in  $a_0$  could induce secular evolution in galaxies and other galactic systems. For example, the mass-velocity relations dictate that in the deep MOND regime the velocities in a system of a given mass should vary like  $a_0^{1/4}$  if they are in the deep MOND regime. This would induce changes in radius, so as to preserve adiabatic invariants, such as possibly  $rv$  (Milgrom 1989b). Such variations in  $a_0$  could also provide an anthropic mechanism for getting  $\Omega_\Lambda \sim 1$  at the present time (Milgrom 1989b, Sanders 1998, 2001).

Analyzing the data of Genzel et al. (2006) on the rotation curve of a galaxy at redshift  $z = 2.38$ , I find that they are consistent with MOND with the local value of  $a_0$ . However, the large error margins, and the fact that at the last measured point the galaxy is only marginally in the MOND regime, still allow appreciable time variations of  $a_0$ .

MOND as it is formulated at present does not provide a clear-cut tool for treating cosmology. In my opinion, the understanding of cosmology in MOND may one day come from the same insight and at the same time as the understanding of the fundamental basis for MOND.

## 7. MOND theories

One can build different detailed theories on the basis of the above premises of MOND. These theories will differ as regards their detailed predictions, but they are expected to share the core predictions listed above, and more like them. For example, in the nonrelativistic regime one can modify the Poisson equation for the gravitational field, or one can modify the kinetic action of particles. Ultimately, one would like to extend the MOND basic premises to the relativistic regime.

There are two general approaches to building MOND theories: The first is to construct actions that incorporate the MOND tenets and thus reproduce MOND phenomenology. (Starting from an action has the usual advantage that it guarantees the standard conservation laws if the action has the standard symmetries.) These have, so far, took  $a_0$  to be a constant of the theory, and usually employ some form of an interpolating

function between the classical and the MOND regimes. The other approach is to start from some idea as to the possible origin of MOND and in this way derive  $a_0$  and  $\mu$  from, or at least relate them to, more basic concepts. At the moment we only have some very preliminary attempts in this vein.

### 7.1. Non-relativistic action formulations

To demonstrate the sort of embodiments of the MOND tenets that are possible in the norelativistic regime we start with the action that describes a Newtonian system of many gravitating point masses

$$S = -\frac{1}{8\pi G} \int d^3r (\vec{\nabla}\phi)^2 - \sum_i m_i \phi(\mathbf{r}_i) + \sum_i m_i \int dt v_i^2(t)/2. \quad (6)$$

The first term is the free action of the gravitational potential field  $\phi$ , the second is the interaction of the masses with the field, and the third is the free (kinetic) action of the particles. We seek to modify this theory so as to incorporate the MOND tenets.

#### 7.1.1. Modified gravity

One possibility is to modify only the free action of the gravitational field. Perhaps the simplest way to do this is to keep the Lagrangian as a function of only the first derivatives of the potential. The most general modification retaining rotational symmetry is then obtained by replacing  $(\vec{\nabla}\phi)^2$  in the first term by  $(\vec{\nabla}\phi)^2 F[(\vec{\nabla}\phi/a_0)^2]$ . As a result the standard Poisson equation for the gravitational potential is replaced by

$$\vec{\nabla} \cdot [\mu(|\vec{\nabla}\phi|/a_0) \vec{\nabla}\phi] = 4\pi G\rho, \quad (7)$$

where  $\mu(\sqrt{y}) \equiv F(1 + d\ln F/d\ln y)$  (Bekenstein & Milgrom 1984). The MOND axioms require that  $F$  goes to 1 ( $\mu$  goes to 1) at large arguments, and  $F$  goes to  $2y^{1/2}/3$  [ $\mu(x)$  goes to  $x$ ] for small arguments. I would classify such a modification as “modified gravity” because it leaves intact the equation of motion of particles in a given field, and it also affects the dynamics only in systems where gravity is important. This theory and its implications for galactic dynamic have been discussed extensively by Bekenstein & Milgrom (1984), Milgrom(1984,1986b), Ciotti & Binney (2004), and others. It has been used in numerical codes to solve for the MOND gravitational field in many-body calculations that were applied to various problems e.g., by Brada & Milgrom (1999b,2000a,b), Tiret & Combes (2007a,b), Nipoti, Londrillo & Ciotti (2007a,b), and others.

All the MOND laws listed in section 2 where shown explicitly to hold in this theory.

#### 7.1.2. Modified inertia

Another possibility (Milgrom 1994a) is to modify the last term in the action of eq.(6), replacing it by a general action of the form

$$\sum_i m_i S_K[a_0, \{\mathbf{r}_i(t)\}]. \quad (8)$$

Here,  $S_k$  is a universal functional of the particle trajectory  $\{\mathbf{r}_i(t)\}$ , independent of the particle’s type. It is possibly non-local, and is a function of  $a_0$ . This modification leaves the Poisson equation intact but changes

the particle equation of motion (Newton’s second law,  $\mathbf{a} = -\vec{\nabla}\phi$ ) into

$$\mathbf{A}[\{\mathbf{r}(t)\}, a_0] = -\vec{\nabla}\phi, \quad (9)$$

where  $\mathbf{A}$  is a (possibly non-local) functional of the trajectory, and a function of  $a_0$ . The dependence on the particle mass is such that the theory automatically preserves the universality of free fall.

MOND tenet (ii) above dictates that for  $a_0 \rightarrow 0$ ,  $S_K \rightarrow \int v^2/2$ , and  $\mathbf{A} \rightarrow \mathbf{a}$ , the acceleration. MOND tenet (iii) dictates that  $S_K[a_0, \{\mathbf{r}_i(t)\}] \rightarrow a_0^{-1} s_K[\{\mathbf{r}_i(t)\}]$ , and so  $\mathbf{A}[\{\mathbf{r}(t)\}, a_0] \rightarrow a_0^{-1} \mathbf{Q}[\{\mathbf{r}(t)\}]$  for  $a_0 \rightarrow \infty$ , where  $\mathbf{Q}$  is a functional of the trajectory with dimensions of acceleration squared. (By multiplying the action by  $a_0^2 G$  and redefining  $a_0 \phi \rightarrow \phi$  we can bring the action to a form where only the product  $m_i G a_0$  appears, as required.)

I showed in Milgrom (1994a) that for such a theory to obey the MOND limits and to be Galilei invariant it has to be non-local, and that this, in fact, has some advantages. Also note that, again, the resulting theory must be nonlinear.

I call such theories modified inertia, as they do not modify the gravitational field, but modify the equations of motion, and this for whatever combination of forces is in action in the system, gravitational or not. As an example, Special relativity would count here as a (time-local, but non-MOND) modification of inertia. It has  $\mathbf{A} = d(\gamma\mathbf{v})/dt = \gamma[\mathbf{a} + \gamma^2(\mathbf{v} \cdot \mathbf{a})\mathbf{v}/c^2]$ .

It was also shown in Milgrom (1994a), as a general result for such theories, that the rotation curves of axisymmetric systems are simply given by eq.(5), where  $\mu(x)$  in this case is derived from the restriction of the kinetic action to circular orbits: When a general action functional is restricted to uniform, circular orbits it reduces to a function of only the radius of the orbit,  $R$ , and the rotational speed,  $V$ . On dimensional grounds, a MOND action that is normalized to have dimensions of velocity squared must then reduce to the form  $I_{circ} = (1/2)V^2 i(V^2/ra_0)$ . Then  $\mu(x) = i(x)[1 + \hat{i}(x)/2]$ , where  $\hat{i} = d \ln i / d \ln x$  (Milgrom 1994a). For a given theory  $\mu(x)$  is the interpolating function that applies only to circular orbits in an axisymmetric potential, but for such orbits it is universal (i.e., unique for the theory, and independent of the exact nature of the potential field). This is indeed the equation that has been used in all MOND rotation curve analyses to date. It follows from this that if the kinetic action contains terms that vanish for circular orbits, such as terms that are proportional to  $(\mathbf{v} \cdot \mathbf{a})^2/a_0^2$  (which is not Galilei invariant), they do not enter  $\mu(x)$ , and thus do not affect circular trajectories. They do, however, affect linear motion as strongly as desired. This is how a modified inertia theory might be constructed that is consistent with planetary motions, while explaining the Pioneer anomaly.

### 7.1.3. Comparison between modified gravity and modified inertia

It has to be realized, in the first place, that the division between the two classes of theories is not always clear, especially so in the relativistic regime. The free matter action in relativity contains the gravitational degrees of freedom in addition to those of matter. This means that modifying the kinetic action will change the gravitational field equations as well as the equation of motion of matter. On the other hand, modifying the gravitational field also modifies the equations of motion. For example, the Brans Dicke theory, which modifies GR, has two equivalent formulations, one in which the action for gravity (the Einstein-Hilbert action) is modified—nominally qualifying as modified gravity—another in which the matter action is modified (modified inertia). But, at least in the nonrelativistic (NR) regime the distinction is rather clear and useful.

We saw in section 2 that many of the major predictions of MOND follow from the basic tenets and are



thus shared by all MOND theories. It is thus not so easy, with present day knowledge, to distinguish even between the two classes of theories. But as there are differences in the predictions of different theories, so there are class differences between the predictions of the two theory types. I mentioned already the fact that in NR modified gravity the gravitational field of a given source distribution is modified, but not the equation of motion; so the acceleration of test particles depends only on their position. In modified inertia the local acceleration depends also on the trajectory of the particle (as in Special Relativity). There is still, in such theories a momentum whose time derivative depends only on position, but this is not the acceleration.

Another difference is in the exact prediction for rotation curves. While for modified inertia eq.(5) is exact, it is only approximate for modified gravity.

Yet another difference is in the definition of the conserved quantities and adiabatic invariants that emerge from the two classes: In modified gravity the usual expressions for kinetic energy and angular momentum hold; not so in modified inertia (a familiar case in point is, again, Special Relativity).

There is, thus, a potential for distinguishing between the two classes from the observations.

## 7.2. Relativistic formulations

A detailed review of relativistic formulations of MOND can be found in Bekenstein (2006), and in Bruneton & Esposito-Farese (2007). The state of the art of this effort is the TeVeS theory (Bekenstein 2004) and its reformulations-generalizations (e.g., Sanders 2005, Zlosnik Ferreira & Starkman 2006,2007), which I now describe very succinctly.

### 7.2.1. TeVeS type theories-description

1. Gravity in TeVeS (for Tensor, Vector, Scalar) is described by a metric  $g_{\alpha\beta}$ , as in GR, plus a vector field,  $U_\alpha$ , and a scalar field  $\phi$ . In the formulation of Zlosnik Ferreira & Starkman (2006) the scalar is eliminated on the expense of adding a degree of freedom in the vector.
2. Matter is coupled to one combination of the fields, dubbed the physical metric
$$\tilde{g}_{\alpha\beta} \equiv e^{-2\phi}(g_{\alpha\beta} + U_\alpha U_\beta) - e^{2\phi} U_\alpha U_\beta$$
3.  $g_{\alpha\beta}$  is governed by the usual Hilbert-Einstein action.
4. The vector field is governed by a Maxwell-like action, but is constrained to have a unit length.
5. The scalar action can be written as  $S_s = -\frac{1}{2Gk\hat{k}} \int Q[\hat{k}(g^{\alpha\beta} - U^\alpha U^\beta)\phi_{,\alpha}\phi_{,\beta}](-g)^{1/2}d^4x$ .
6. There are three constants appearing in this formulation:  $k$ ,  $\hat{k}$ , and  $K$ , and one free function  $Q(x)$ . The last begets the interpolating function of MOND in the nonrelativistic limit.

### 7.2.2. TeVeS type theories-results

1. For nonrelativistic Galactic systems it reproduces the nonrelativistic MOND phenomenology yielding for this case eq.(7) with  $a_0 \propto k\hat{k}^{-1/2}$ .

2. Lensing in weak fields ( $\phi \ll c^2$ ): TeVeS gives lensing according to the standard GR formula but with the MOND potential derived from eq.(7).
3. Structure formation: Preliminary work is described in Sanders (2005), Dodelson & Liguori (2006), Skordis (2006), and Skordis et al. (2006).
4. CMB: preliminary work: TeVeS has the potential to mimic aspects of cosmological DM (Skordis et al. 2006).
5. Desiderata:  $a_0$  and the interpolating function are still put in by hand.

### 7.3. Effective theories

The coincidence of  $a_0$  with cosmic acceleration parameters hints at the possibility that MOND is an effective, or emergent, theory. This would mean that MOND might emerge as an approximate consequence of some deeper physical theory. In such a scheme  $a_0$  might turn out not to be a fundamental constant of the underlying theory. This is similar to the case of the free fall acceleration on earth, which appears as a constant of nature when we deal with dynamics near the surface of the earth, but has no significance in the underlying gravitation theory. Likewise, the interpolating function, or its equivalents, that appear in MOND theories should be derivable from the underlying theory. The role of this function in MOND is similar to that of the black body function in quantum mechanics, or to that of the Lorentz factor in Special Relativity. They too interpolate between the appropriate classical limit and the modified regime, with the fundamental constant of the theory setting the boundary. They too were introduced first on phenomenological grounds, but were then derived from a basic underlying theory.

For example, MOND could result from the imposition of a new symmetry generalizing Lorentz invariance (Milgrom 2005) in a way that connects cosmology with local physics. There are several ideas for possible underlying schemes for MOND. All are at the moment rather preliminary. I list three of these below.

#### 7.3.1. Vacuum effects

Applying MOND as it is now formulated requires knowledge of some inertial frame with respect to which absolute accelerations can be measured: Because MOND is nonlinear we need to substitute in its equations a value of the absolute acceleration. One usually assumes that this is the local rest frame of galaxies—the cosmologically comoving frame (e.g., in applications of the external field effect). But what could be the physics underlying the choice of such a frame? Milgrom (1999) pointed out that the quantum vacuum might provide it. The vacuum constitutes a physical inertial frame in the sense that any observer with some internal structure can detect its own non inertial motion with respect to it. This can be done via the Unruh effect by which a non inertial observer in an otherwise empty Minkowski vacuum detects Unruh radiation that depends in a complicated way on the observer’s world-line. For the simplest, nontrivial case of an observer with a constant acceleration,  $a$ , the radiation is thermal with a temperature  $T = \alpha a$ ,  $\alpha \equiv \hbar/2\pi k c$ . This, or something like it, could be the sensor that tells a system that it is being accelerated, and thus be the marker for inertia. An analog effect takes place even for an inertial observer in an expanding universe. Again, in the simplest case of a de Sitter universe an inertial observer finds itself immersed in thermal radiation with  $T = \alpha c(\Lambda/3)^{1/2}$ , where  $\Lambda$  is the appropriate cosmological constant defining the curvature of space time (the

Gibbons-Hawking effect). This may somehow be felt by bodies and imprint an effect of cosmology in local dynamics, as is hinted by MOND. An observer with a constant acceleration world line in a de Sitter universe also senses thermal radiation with temperature  $T = \alpha(a^2 + c^2\Lambda/3)^{1/2}$  (Narnhofer, Peter, & Thirring 1996, Deser & Levin 1997). The temperature difference between such an observer and one that is inertial is then

$$\Delta T = \alpha[(a^2 + c^2\Lambda/3)^{1/2} - c(\Lambda/3)^{1/2}] = \alpha a \mu(a/\hat{a}_0), \quad (10)$$

with  $\mu(x) = [(1 + 4x^2)^{1/2} - 1]/2x$ , and  $\hat{a}_0 = 2c(\Lambda/3)^{1/2}$ . The two limits of this temperature difference for high and low accelerations are:

$$\Delta T \propto \begin{cases} a & : a \gg \hat{a}_0 \\ a^2/\hat{a}_0 & : a \ll \hat{a}_0 \end{cases}. \quad (11)$$

This is very much reminiscent of what is required for MOND inertia, with the added bonus of a relation between  $\hat{a}_0$  and the cosmological parameter  $\Lambda$ . Applied to the Pioneer spacecraft, using the measured value of the cosmological constant, such an expression for inertia gives  $a = MG/r^2 + \hat{a}_0/2$ , with  $\hat{a}_0/2 \approx 6 \times 10^{-8} \text{ cm s}^{-2}$ , compared with the measured value of the anomaly  $a_{an} \approx 8 \times 10^{-8} \text{ cm s}^{-2}$ . Note that we do not live exactly in a de Sitter universe and that the motion of the Pioneer spacecraft is not exactly one of constant acceleration, for which situation the above expression was derived.

Note also that for modified inertia, as would be the case here, we cannot, without a theory, learn much about circular orbits from observations of linear ones like those of the Pioneer spacecraft—with acceleration parallel to the velocity. In particular the effective interpolating function applicable to rotation curves need not be the same as that for linear trajectories that we find here. Also, the fact that  $\hat{a}_0$  here isn't exactly  $a_0$  we find for rotation curves isn't necessarily meaningful. This too would just point to a different effective  $\mu(x)$  that behaves as  $\lambda x$  at small  $x$  (with  $\lambda \approx .1$ ), not as  $x$  as in the case of circular orbits. All this should be expected: if the Pioneer anomaly is at all a MOND effect there will have to be a very different effect on quasi circular orbits since, as I mentioned above, the anomaly is not detected for the planets.

### 7.3.2. Membrane models

Equation (7) is identical in form to the equation that determines the shape of a membrane that tries to minimize its area (or its volume, in higher dimensions). The potential  $\phi$  then stands for the height of this membrane above some reference plane. (See, e.g., Milgrom 2002c for a discussion of this and other physical systems governed by an equation of the same form.) However, the resulting  $\mu$  function is not that of MOND. Milgrom (2002a) discussed a scenario in which our 3-space is a 3-D membrane moving in a 4-D space. In this last there is a preferred direction, in which the membrane is being accelerated. The position along this direction is perceived on the membrane as the gravitational potential (all this is a nonrelativistic description in need of a relativistic extension). The energy function of the membrane is not simply its area but one that depends also on the orientation of the membrane with respect to the preferred direction.

### 7.3.3. Polarizable medium

Equation(7) is also identical in form to the equation for the electrostatic potential in a nonlinear, dielectric medium in which the dielectric constant is a function of the field strength (e.g., Milgrom 2002c). This has lead Blanchet (2007a,b, see also this volume) to propose a physical theory for MOND based on a gravitationally polarizable medium.

## 8. Significance for DM

It is sometimes claimed by DM advocates that the successes of MOND will one day be understood in terms of DM, meaning that MOND somehow summarize how DM acts. This cries for undeceiving: Can the myriad observations on the distribution of “DM” be all gotten from the baryon distribution alone through a very simple formula involving one universal parameter? Can the ubiquitous appearance of the constant  $a_0$  in seemingly independent galactic phenomena emerge somehow from the DM paradigm?

The nature and origin of mass discrepancies in galactic systems differ greatly in MOND and in the Newtonian-dynamics-plus-DM paradigms. In MOND, these discrepancies are not real, they are artifacts of adhering to Newtonian dynamics instead of MOND. As a result, MOND predicts them uniquely from the presently observed (baryonic) mass distribution. As we saw, the pattern of these discrepancies is predicted, and is observed, to follow a large number of well defined relations. For MOND to be some summarizing DM formula we will have to conclude that the distribution of baryons fully determines that of the DM. However, in the DM paradigm the expected distributions of the two components are strongly dependent on details of the particular history of a system: The two types of matter are very different: baryons are dissipative and strongly interacting with photons and magnetic fields; CDM supposedly is neither. Along the haphazard history of a galactic system they are then subject to different influences. The formation process and the ensuing unknown and *unknowable* history of mergers, cannibalism, gas accretion, ejection of baryons by supernovae and ram pressure, energy loss by dissipation, interaction with magnetic fields, etc., all affect baryons and DM differently. These processes are expected to produce haphazard relative amounts and distributions of the two components in the system. The fact that baryons and DM are well separated today, and that baryons form discs while CDM does not are obvious results of such differentiation. Another strong and direct evidence comes from the recent realization that the ratio of baryons to required DM in present day galaxies is smaller by an order of magnitude, typically, than the cosmic value, with which protogalaxies presumably started their life. This evidence comes, for example, from probing large galactic radii with weak lensing ; e.g. by Kleinheinrich et al. (2004), Mandelbaum et al. (2005), and by Parker et al. (2007) (see also McGaugh 2007 for evidence based on small radius data with CDM modeling). This means that in the DM paradigm, galaxies should have somehow lost most of their baryons ( $\sim 90\%$ ) during their history. Even for galaxy clusters there is now some tentative evidence that the observed baryon fraction is a few tens of percents smaller than the cosmic value (Afshordi et al. 2007 and references therein).

Another poignant example of the large variety of baryon vs. DM properties expected in the DM paradigm is brought into focus by the recent observation of large mass discrepancies in three tidal debris dwarf galaxies (Bournaud et al. 2007). This case is doubly interesting because it is also one where CDM and MOND predictions differ greatly. In light of the specific formation scenario of such dwarfs, CDM predicts very small amount of DM in them (see discussion of Bournaud et al.). This is in contrast with what is expected in primordial dwarfs for which large DM to baryon fractions are predicted. Since both types of dwarfs are low acceleration systems, MOND predict large mass discrepancies in both. The dwarfs analyzed by Bournaud et al. (2007) do show substantial mass discrepancies as predicted by MOND as shown in Fig. 8 (Milgrom 2007, Gentile et al. 2007b), and contrary to what CDM predicts.

In a DM interpretation of MOND one will then conclude that the haphazard and small amount of leftover baryons in galaxies determine so many of the properties of the dominant DM halo, with such accuracy as evidenced by MOND. I deem it highly inconceivable that DM will ever reproduce the predictions of MOND and the relations it predicts for individual systems. Achieving this within the complex scenario of galaxy formation would be akin to predicting the detailed properties of a planetary system—planet masses, radii,

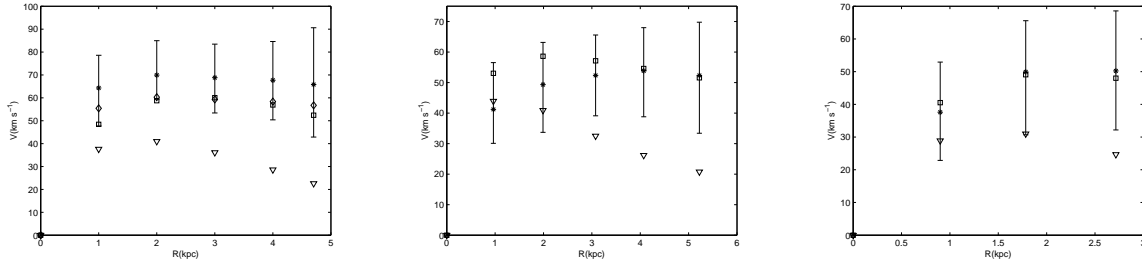


Fig. 8.— The rotation curves for the three debris dwarfs from Bournaud et al. (2007): NGC 5291N (left), 5291S (center), and 5291SW (right). The measured velocities, for the nominal inclination of  $45^\circ$ , are marked by stars and are shown with their error bars. The calculated Newtonian velocities are marked by inverted triangles. The predicted MOND velocities are marked by squares. Also shown, for NGC 5291N only, the measured velocities for an assumed inclination of  $i = 55^\circ$  (diamonds); from Milgrom (2007).

and orbits—from knowledge of only the present day central star. When we find a tight relation between system properties in astronomy—such as the zero-age main sequence for non-rotating, nonmagnetic stars of a given composition—it results inescapably from laws of physics. We cannot conceive of an incipient star (non-rotating, non-magnetic, etc.) that does not sit on the main sequence; we cannot conceive of a galaxy in MOND that does not satisfy the MOND laws; but we can easily conceive of such a galaxy in the Newtonian dynamics plus DM paradigm.

Indeed, to my knowledge none of the MOND predictions listed in section 2 has been shown to follow in the DM paradigm. (I discount the attempt by Kaplinghat and Turner 2002 to explain prediction 3, for the reasons I gave in Milgrom 2002b.)

For example, it is being claimed that LCDM predicts a Tully-Fisher relation not unlike what is observed. This, however, is not really true. What LCDM predicts is a relation between the total halo mass and its maximum circular-orbit velocity. In earlier times it could be assumed with impunity that the ratio of total baryons to total DM mass in galaxies is universal (and equal to the cosmological value). This would have given a relation between the baryon mass and the rotational velocity for the halo. However, the assumption of a universal baryon fraction is now known not to follow in the CDM picture, and, as we saw, this fraction is typically much smaller than the putative cosmological value. Not only isn’t the general correlation predicted, but the processes that caused only a small fraction of the original baryons to show up in present day galaxies are likely to have produced large scattering in this ratio, and hence in the predicted baryon-mass-velocity relation, unlike what is observed. (Simulations that include baryons are very ad hoc, and introduce many assumptions by hand. Clearly, true simulations of baryon behavior in this context are far beyond the offing.)

Regarding the ability of DM models to fit rotation curves of disc galaxies: Apart from the well known lingering problem of fitting the inner parts of galactic rotation curves (the “cusp problem”), halos predicted by LCDM (with, e.g., NFW profiles) can, by and large, fit the observed curves, but only if one leaves free their mass and size parameters. Such fits then involve three parameters: the stellar mass-to-light ratio (the only parameter used in MOND fits) plus the two structural parameters for the halo, which afford great freedom in reconstructing rotation curves. However, in LCDM simulations the two structural parameters are not free, but come out strongly correlated. If one actually uses the halo profiles predicted by LCDM, with that correlation enforced, the rotation curve fits are bad, even though they still have the freedom of

an additional halo parameter vis-a-vis MOND (e.g., McGaugh et al. 2007, Gentile Tonini & Salucci 2007, Gentile et al. 2007a).

Dark matter advocates themselves invoke “baryon-less” galaxies when they want to explain the “missing satellite” problem for CDM. But this would then contradict the existence of strict relations between baryons and DM, which they would have to invoke to explain MOND predictions within the DM paradigm. The blanket is too short to cover both ends.

To recapitulate, the confirmations of the MOND predictions—e.g., those listed in section 2 and those concerning the full rotation curves—argue against the Newtonian-dynamics-plus-DM paradigm in two ways. First, since they supports MOND as a competing paradigm. Second, because in themselves, and without reference to MOND, these regularities point to a strong baryon-DM connection in which Baryons determine completely the distribution of DM in a many well defined and independent ways, and object by object. This would clearly fly in the face of the expectations from DM in which the relation between baryons and DM is haphazard and strongly dependent on the history of each object.

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